J Glob Optim (2006) 36:499–516
DOI 10.1007/s10898-006-9023-2

ORIGINAL ARTICLE

Duality for Multiple Objective Fractional Subset Programming with Generalized (Ƒ, ρ, σ, θ)-V-Type-I Functions*

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Received: 4 November 2004 / Accepted: 8 March 2006 /
Published online: 14 June 2006

Abstract In this paper, we use a new class of generalized convex n-set functions, called (Ƒ, ρ, σ, θ)-V-Type-I and related non-convex functions to establish appropriate duality theorems for three parametric and three semi-parametric dual models to the primal problem. This work extends an earlier work of Zalmai [Computer and Mathematics with Applications 43 (2002) 1489–1520] to a wider class of functions.

Keywords Multiple Objective fractional subsets programming · Generalized n-set convex functions · Duality

1 Introduction

Consider the following multiple objective fractional subset programming problem:

$$\text{Minimize: } \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)},\right) \tag{P}$$
$$\text{Subject to: } H_j(S) \leqq 0,\ j \in \underline{m},\ S \in \Lambda^n,$$

where $\Lambda^n$is the n-fold product of the σ-algebra $\Lambda$of the subsets of a given set X, F_i, G_i, $i \in \underline{p} \equiv \{1, 2, \ldots, p\}$ $H_j(S) \leqq 0$, $j \in \underline{m} \equiv \{1, 2, \ldots, m\}$, are real valued functions defined on Λ^n, and for each $G_i(S) > 0$, for each $i \in \underline{p}$, for all $S \in \Lambda^n$.

The point-function counterparts of (P) are known in the area of mathematical programming as multiple objective fractional programming problems. These problems have been the focus of intense interest in the past few years, which has resulted in numerous publications

*This research is supported by the Department of Science and Technology, Ministry of Science and Technology, Government of India, under the Fast Track Scheme for Young Scientist through grant No. SR/FTP/MS-22/2001.

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the reader may consult a fairly extensive list of references related to various aspects of fractional programming in [29]. For more information about general multiobjective problems with point-functions, the reader may consult [4, 26, 27].

In the area of subset programming, multiobjective problems have been investigated in [15, 18], and multiobjective fractional problems in [15–17]. Much attention has been paid to the analysis of optimization problems with set functions, for example, see Chou et al. [1], Corley [2], Hsia and Lee [10, 11], Hsia et al. [12], Kim et al. [18], Lin [20–23], Liu [24], Mazzoleni [25], Morris [28], Preda [30, 31], Preda and Minasian [32–34], and Zalmai [39–41]. A formulation for optimization problems with set functions was first given by Morris [28]. The main results of Morris [28] are confined only to set functions of a single set. Corley [2] gave the concepts of a partial derivative and a derivative of real-valued n-set functions. Chou et al. [1], Kim et al. [17, 18], Lai and Lin [19], Lin [20–23], Preda [30, 31], and Preda and Minasian [32–34] studied optimality and duality for optimization problems involving vector-valued n-set functions. For details, one can refer to Hsia and Lee [10, 11], Hsia et al. [12], Kim et al. [17, 18], Lin [20–23], Mazzoleni [25], Mishra [27], Preda [30], Rosenmuller and Weidner [35], Tanaka and Maruyama [37] and Zalmai [39–42].

Zalmai [42] introduced a new class of generalized convex n-set functions and then presented a number of parametric and semi-parametric sufficient efficiency conditions under the assumptions introduced in [42] for a multiobjective fractional subset programming problem. Moreover, three parametric and three semi-parametric dual models are given and appropriate duality results are established under the aforesaid assumptions, in [42].

Starting from the methods used by Craven [3], Giorgi and Molho [5], Hachimi and Aghezzaf [6], Hanson [7], Hanson and Mond [8], Jeyakumar [13] and Jeyakumar and Mond [14], Mishra [26], Rueda and Hanson [36] and Ye [38], Preda and Minasian [34] defined some new classes of scalar and vector functions called d-type-I, d-pseudo type-I and d-quasi type-I for a multiobjective programming problem involving n-set functions and obtained a few interesting results on optimality and Wolfe duality.

Recently, Hanson et al. [9] introduced a new class of functions called vector type-I and its generalizations. Motivated by Hanson et al. [9], In this paper, we extend the work of Zalmai [42] to the class of V-type-I and related functions. We present six dual models for (P). Three parametric models whose forms and properties are based on the Theorems 2.1, 3.1–3.3 [42]; and three semi-parametric models whose structure and contents are motivated by Theorems 2.2, 4.1–4.3 [42]. In each case, we establish appropriate weak and strong duality theorems. The paper is organized as follows. In Section 2, we recall the definitions of differentiability, convexity and certain type of generalized convexity Type-I and related functions for n-set functions, which will be used frequently throughout the sequel. In Section 3, we consider a simple dual problem and prove weak and strong duality theorems under generalized (F, ρ, σ, θ)-V-Type-I and related non-convex functions for a parametric dual model for (P). In Sections 4 and 5, we formulate two general parametric dual models that are, in fact, two families of dual problems whose members can readily be identified by appropriate choices of certain sets and functions. In Sections 6–8, we discuss the semi-parametric counterparts of the dual models presented in Sections 3–5.

Notice that, all these results are also applicable, when appropriately specialized, to the following three classes of problems with multiple, fractional, and conventional objective functions, which are particular cases of (P):

$$\underset{S\in X}{\text{Minimize}} \left(F_1(S),\ F_2(S), \ldots, F_p(S)\right) \tag{P1}$$

$$\underset{S\in X}{\text{Minimize}} \frac{F_1(S)}{G_1(S)} \tag{P2}$$

$$\underset{S\in X}{\text{Minimize}}\, F_1(S) \tag{P3}$$

where X(assumed to be nonempty) is the feasible set of (P), that is,

$$X = \left\{S \in \Lambda^n : H_j(S) \leqq 0,\ j \in \underline{m}\right\}.$$

2 Preliminaries

In this section we gather, for convenience of reference, a number of basic definitions that will be used often throughout the sequel, and recall some auxiliary results.

Let (X, Λ, μ) be a finite atomless measure space with $L_1(X, \Lambda, \mu)$ separable, and let d be the pseudometric on Λ^n defined by

$$d(R, S) = \left[\sum_{i=1}^{n} \mu^2(R_i \Delta S_i)\right]^{1/2},\ R = (R_1, R_2, \ldots, R_n),\ S = (S_1, S_2, \ldots, S_n) \in \Lambda^n,$$

where Δ denotes the symmetric difference; thus, (Λ^n, d) is a pseudo-metric space. For $h \in L_1(X, \Lambda, \mu)$ and $T \in \Lambda$ with characteristic function $\chi_T \in L_\infty(X, \Lambda, \mu)$, the integral $\int_T h\mathrm{d}\mu$ will be denoted by $\langle h, \chi_T\rangle$.

We next recall the notion of differentiability and convexity for n-set functions. They were originally introduced by Morris [28] for set functions, and subsequently extended by Corley [2] for n-set functions.

Definition 2.1 A function $F: \Lambda \to R$ is said to be differentiable at S^* if there exists $DF(S^*) \in L_1(X, \Lambda, \mu)$,called the derivative of F at S^*, such that for each $S \in \Lambda$,

$$F(S) = F\left(S^*\right) + \left\langle \mathrm{DF}\left(S^*\right),\ \chi_S - \chi_{S^*}\right\rangle + V_F\left(S,\ S^*\right),$$

where $V_F(S,\ S^*)$ is $o(d(S,\ S^*))$, that is, $\lim_{d(S,S^*)\to 0} V_F(S,\ S^*)\big/d(S, S^*) = 0$.

Definition 2.2 A function $G:\ \Lambda^n \to R$ is said to have a partial derivative at $S^* = \left(S_1^*, S_2^*, \ldots, S_n^*\right) \in \Lambda^n$ with respect to its ith argument if the function $F(S_i) = G\left(S_1^*, \ldots, S_{i-1,}^*, S_i^*, S_{i+1}^*, \ldots, S_n^*\right)$ has derivative $DF\left(S_i^*\right), i \in \underline{n}$; in that case, the ith partial derivative of G at S^* is defined to be $D_i G(S^*) = DF\left(S_i^*\right), i \in \underline{n}$.

Definition 2.3 A function $G:\ \Lambda^n \to R$ is said to be differentiable at S^* if all the partial derivatives $D_i G(S^*), i \in \underline{n}$ exist and

$$G(S) = G\left(S^*\right) + \sum_{i=1}^{n} \left\langle DG_i\left(S^*\right),\ \chi_{S_i} - \chi_{S_i^*}\right\rangle + W_G\left(S,\ S^*\right),$$

where $W_G(S,\ S^*)$ is $o(d(S,\ S^*))$, for all $S \in \Lambda^n$.

It was shown by Morris [28] that for any triplet $(S, T, \lambda) \in \Lambda \times \Lambda \times [0,\ 1]$, there exist sequences $\{S_k\}$ and $\{T_k\} \in \Lambda$ such that

$$\chi_{S_k} \xrightarrow{w^*} \lambda\chi_{S\setminus T} \quad \text{and} \quad \chi_{T_k} \xrightarrow{w^*} \lambda\chi_{T\setminus S} \tag{2.1}$$

imply

$$\chi_{S_k \cup T_k \cup (S\cap T)} \xrightarrow{w^*} \lambda\chi_S + (1-\lambda)\chi_T, \tag{2.2}$$

where $\overset{w^*}{\to}$ denotes the weak* convergence of elements in $L_\infty(X, \Lambda, \mu)$ and $S \backslash T$ is the complement of T relative to S. The sequence$\{V_k(\lambda)\} = \{S_k \cup T_k \cup (S \cap T)\}$ satisfying (2.1) and (2.2) is called the Morris sequence associated with $(S,\ T,\ \lambda)$.

It was shown in [2, 28] that if a differentiable function $F\colon \Lambda \to R$ is convex, then

$$F(S) \geqq F(T) + \sum_{i=1}^{n} \left\langle D_i F(T), \chi_{S_i} - \chi_{T_i} \right\rangle, \quad \forall\, S, T \in \Lambda^n.$$

Following the introduction of the notion of convexity for set functions by Morris [28] and its extension for n-set functions by Corley [2], various generalizations of convexity for set and n-set functions were proposed in [33].

For predecessors and point-function counterparts of these convexity concepts, the reader is referred to the original papers where the extensions to set and n-set functions are discussed. A survey of recent advances in the area of generalized convex functions and their role in developing optimality conditions and duality relations for optimization problems is given in [29].

For the purpose of formulating several dual models for (P) and proving various duality results, in this study we shall use a new class of generalized convex n-set functions, called (Ƒ, ρ, σ, θ)-V-Type-I and related non-convex functions, that will be defined later in this section. This class of functions may be viewed as an n-set version of a combination of three classes of point-functions: Ƒ-convex functions, type-I functions and V-invex functions, which were introduced in [7, 14, 37].

Let $S, S^* \in \Lambda^n$, let the function $F\colon \Lambda^n \to R^p$, with components $F_i, i \in \underline{p}$, be differentiable at S^*, let Ƒ(S,S*; .) : $L_1^n(X, \Lambda, \mu) \to R$ be a sublinear function, and let $\theta\colon \Lambda^n \times \Lambda^n \to \Lambda^n \times \Lambda^n$ be a function such that $S \neq S^* \Rightarrow \theta(S, S^*) \neq (0, 0)$.

Definition 2.4 The pair of functions $(F,\ G)$are said to be (Ƒ, ρ, σ, θ)-V-type-I at S^*if there exist functions $\alpha_i\colon \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\},\ i \in \underline{p}, \beta_j\colon \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\},\ j \in \underline{m}, \rho \in R$ and $\overline{\rho} \in R$ such that for each $S \in \Lambda^n$, $i \in \underline{p}$ and $j \in \underline{m}$,

$$F_i(S) - F_i(S^*) \geqq Ƒ\left(S, S^*; \alpha_i(S, S^*) DF_i(S^*)\right) + \rho d^2\left(\theta(S, S^*)\right)$$

and

$$-G_j(S^*) \geqq Ƒ\left(S, S^*; \beta_j(S, S^*) DG_j(S^*)\right) + \bar{\rho} d^2\left(\theta(S, S^*)\right).$$

Definition 2.5 The pair of functions $(F,\ G)$are said to be (Ƒ, ρ, σ, θ)-V-pseudo-quasi-type-I at S^*if there exist functions $\alpha_i\colon \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\},\ i \in \underline{p}, \beta_j\colon \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\}$, $j \in \underline{m}$, $\rho \in R$ and $\overline{\rho} \in R$ such that for each $S \in \Lambda^n$, $i \in \underline{p}$ and $j \in \underline{m}$,

$$Ƒ\left(S, S^*; \sum_{i=1}^{p} DF_i(S^*)\right) \geqq -\rho d^2\left(\theta(S, S^*)\right) \Rightarrow \sum_{i=1}^{p} \alpha_i(S, S^*) F_i(S) \geqq \sum_{i=1}^{p} \alpha_i(S, S^*) F_i(S^*)$$

and

$$-\sum_{j=1}^{m} \beta_j(S, S^*) G_j(S^*) \leqq 0 \Rightarrow Ƒ\left(S, S^*; \sum_{j=1}^{m} DG_j(S^*)\right) \leqq -\bar{\rho} d^2\left(\theta(S, S^*)\right).$$

Definition 2.6 The pair of functions $(F,\ G)$are said to be (Ƒ, ρ, σ, θ)-V-quasi-pseudo-type-I at S^*if there exist functions $\alpha_i\colon \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\},\ i \in \underline{p}, \beta_j\colon \Lambda^n \times \Lambda^n \to R_+ \backslash \{0\},\ j \in \underline{m}$, $\rho \in R$ and $\overline{\rho} \in R$ such that for each $S \in \Lambda^n$, $i \in \underline{p}$ and $j \in \underline{m}$,

$$\sum_{i=1}^{p}\alpha_i\left(S,S^*\right)F_i\left(S\right)\leqq\sum_{i=1}^{p}\alpha_i\left(S,S^*\right)F_i\left(S^*\right)\Rightarrow \digamma\left(S,S^*;\sum_{i=1}^{p}DF_i\left(S^*\right)\right)\leqq-\rho d^2\left(\theta\left(S,S^*\right)\right)$$

and

$$\digamma\left(S,S^*;\sum_{j=1}^{m}Dg_j\left(S^*\right)\right)\geqq-\bar{\rho}d^2\left(\theta\left(S,S^*\right)\right)\Rightarrow-\sum_{j=1}^{m}\beta_j\left(S,S^*\right)G_j\left(S^*\right)\geqq 0.$$

Throughout this paper, we shall deal exclusively with efficient solutions of (P). We recall that an $S^* \in \Xi$ is said to be an efficient solution of (P) if there is no $S \in \Xi$ such that

$$\left(\frac{F_1\left(S\right)}{G_1\left(S\right)},\frac{F_2\left(S\right)}{G_2\left(S\right)},\ldots,\frac{F_p\left(S\right)}{G_p\left(S\right)}\right)\leq\left(\frac{F_1\left(S^*\right)}{G_1\left(S^*\right)},\frac{F_2\left(S^*\right)}{G_2\left(S^*\right)},\ldots,\frac{F_p\left(S^*\right)}{G_p\left(S^*\right)}\right).$$

In order to derive a set of necessary conditions for (P), we employ a Dinkelbach-type [4] indirect approach via the following auxiliary problem:

$$(\mathrm{P}\lambda)\qquad \underset{S\in\Xi}{\text{Minimize}}\left(F_1\left(S\right)-\lambda_1G_1\left(S\right),\ldots,F_p\left(S\right)-\lambda_pG_p\left(S\right)\right),$$

where $\lambda_i,\ i\in\underline{p}$, are parameters. This problem is equivalent to (P) in the sense that for particular choices of $\lambda_i,\ i\in\underline{p}$, the two problems have the same set of efficient solutions. This equivalence is stated more precisely in the following lemma whose proof is straightforward, and hence, omitted.

Lemma 2.1 *An $S^*\in\Xi$ is an efficient solution of (P) if and only if it is an efficient solution of (Pλ^*) with $\lambda_i^*=F_i\left(S^*\right)/G_i\left(S^*\right),\ i\in\underline{p}$.*

Now applying Theorem 3.23 of [22] to (Pλ) *and using Lemma 2.1, we obtain the following necessary efficiency results for (P).*

Theorem 2.1 *Assume that $F_i, G_i, i\in\underline{p}$, and $H_j,\ j\in\underline{m}$, are differentiable at $S^*\in\Lambda^n$, and that for each $i\in\underline{p}$, there exists $\hat{S}^i\in\Lambda^n$ such that*

$$H_j\left(S^*\right)+\sum_{k=1}^{n}\left\langle D_kH_j\left(S^*\right),\chi_{\hat{S}_k^i}-\chi_k^*\right\rangle<0,\ \ j\in\underline{m},$$

and for each $l\in\underline{p}\backslash\{i\}$,

$$\sum_{k=1}^{n}\left\langle D_kF_l\left(S^*\right)-\lambda_l^*D_kG_l\left(S^*\right),\chi_{\hat{S}_k^l}-\chi_{S_k^*}\right\rangle<0.$$

If S^* is an efficient solution of (P) and $\lambda_i^*=F_i\left(S^*\right)/G_i\left(S^*\right),\ i\in\underline{p}$, then there exist $u^*\in U=\left\{u\in R^p\colon u>0,\sum_{i=1}^{p}u_i=1\right\}$ and $v^*\in R_+^m$ such that

$$\sum_{k=1}^{n}\left\langle\sum_{i=1}^{p}u_i^*\left[D_kF_i\left(S^*\right)-\lambda_i^*D_kG_i\left(S^*\right)\right]+\sum_{j=1}^{m}v_j^*D_kH_j\left(S^*\right),\chi_{S_k}-\chi_{S_k^*}\right\rangle\geqq 0,\forall\ S\in\Lambda^n,$$

$$v_j^*H_j\left(S^*\right)=0,\ \ j\in\underline{m}.$$

The above theorem contains two sets of parameters u_i^* and $\lambda_i^*,\ i\in\underline{p}$. It is possible to eliminate one of these two sets of parameters, and thus, obtain a semi-parametric version of Theorem 2.1. Indeed, this can be accomplished by simply replacing λ_i^* by $F_i\left(S^*\right)/G_i\left(S^*\right)$, $i\in\underline{p}$, and redefining u^* and v^*. For further reference, we state this in next theorem.

Theorem 2.2 *Assume that $F_i, G_i, i \in \underline{p}$, and H_j, $j \in \underline{m}$, are differentiable at $S^* \in \Lambda^n$, and that for each $i \in \underline{p}$, there exists $\hat{S}^i \in \Lambda^n$ such that*

$$H_j\left(S^*\right) + \sum_{k=1}^{n} \left\langle D_k H_j\left(S^*\right), \chi_{\hat{S}^i_k} - \chi^*_k \right\rangle < 0, \quad j \in \underline{m},$$

and for each $l \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \left\langle G_l\left(S^*\right) D_k F_l\left(S^*\right) - F_l\left(S^*\right) D_k G_l\left(S^*\right), \chi_{\hat{S}^l_k} - \chi_{S^*_k} \right\rangle < 0.$$

If S^* is an efficient solution of (P), then there exist $u^* \in U$ and $v^* \in R^m_+$ such that

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} u^*_i \left[G_i\left(S^*\right) D_k F_i\left(S^*\right) - F_i\left(S^*\right) D_k G_i\left(S^*\right)\right] + \sum_{j=1}^{m} v^*_j D_k H_j\left(S^*\right), \chi_{S_k} - \chi_{S^*_k} \right\rangle$$
$$\geqq 0, \forall S \in \Lambda^n, v^*_j H_j\left(S^*\right) = 0, \quad j \in \underline{m}.$$

The form and contents of the necessary efficiency conditions given in Theorem 2.2 are used by Zalmai [42] to derive a number of semi-parametric sufficient efficiency criteria as well as for constructing various dual models for (P).

3 Dual model I

In this section, we consider the following dual problem:

$$\text{(DI)} \qquad \text{Maximize}\, \lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_p\right)$$

subject to

$$\digamma\left(S, T; \sum_{i=1}^{p} u_i \left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0, \quad \forall S \in \Lambda^n, \quad (3.1)$$

$$u_i \left\lfloor F_i(T) - \lambda_i G_i(T)\right\rceil \geqq 0, \quad i \in \underline{p}, \quad (3.2)$$

$$v_j H_j(T) \geqq 0, \quad j \in \underline{m}, \quad (3.3)$$

$$T \in \Lambda^n, \ \lambda \in R_+, \ u \in U, \ v \in R^m_+,$$

where $\digamma(S, T; \cdot) : L^n_1(X, \Lambda, \mu) \to R$ is a sublinear function. Throughout our discussion, we assume that the functions $F_i, G_i, i \in \underline{p}$, and H_j, $j \in \underline{m}$, are differentiable on Λ^n. We shall introduce along the way some additional notations. For stating our first duality theorem, we use the real-valued functions $\mathrm{A}_i(\cdot\,; \lambda, u)$ and $\mathrm{B}_j(\cdot\,, v)$ defined for fixed λ, u and v on Λ^n by

$$\mathrm{A}_i(\cdot\,; \lambda, u) = u_i\left[F_i(S) - \lambda_i G_i(S)\right], \quad i \in \underline{p},$$

and

$$\mathrm{B}_j(\cdot\,, v) = v_j H_j(S), \quad j \in \underline{m}.$$

Theorem 3.1 *Let S and (T, λ, u, v) be arbitrary feasible solutions of (P) and (DI), respectively, and assume that any one of the following sets of hypotheses is satisfied:*

(a) *(i)* $\left(\mathrm{A}_i\left(\cdot\,;\lambda,u\right),\ \mathrm{B}_j\left(\cdot,v\right)\right)$ $\forall\ i\ \in\ \underline{p}$ *and* $j\ \in\ \underline{m},$ *are* $(\mathrm{F},\ \alpha,\ \beta,\ \rho,\ \sigma,\ \theta)$*-V-pseudo-quasi-type-I at* T;
(ii) $\rho+\sigma \geqq 0$;

(b) *(i)* $\left(\mathrm{A}_i\left(\cdot\,;\lambda,u\right),\ \mathrm{B}_j\left(\cdot,v\right)\right)$ $\forall\ i\in \underline{p}$ *and* $j\in\underline{m},$ *are* $(\mathrm{F},\alpha,\beta,\rho,\sigma,\theta)$*-V-pseudo-pre-strict-quasi-type-I at* T;
(ii) $\rho+\sigma > 0$;

(c) *(i)* $\left(\mathrm{A}_i\left(\cdot;\lambda,u\right),\ \mathrm{B}_j\left(\cdot,v\right)\right)\forall\, i\in\underline{p}$ *and* $j\in\underline{m},$ *are* $(\mathrm{F},\alpha,\ \beta,\rho,\sigma,\theta)$*-V-prestrict-quasi-strict-pseudo-type-I at* T;
(ii) $\rho+\sigma \geqq 0$.

Then, $\phi\left(S\right)\equiv\left(\frac{F_1\left(S\right)}{G_1\left(S\right)},\frac{F_2\left(S\right)}{G_2\left(S\right)},\ldots,\frac{F_p\left(S\right)}{G_p\left(S\right)}\right)\nleq\lambda$.

Proof Let S be an arbitrary feasible solution of (P), then, by the sublinearity of F and (3.1), it follows that

$$\mathrm{F}\left(S,T;\sum_{i=1}^{p}u_i\left[DF_i\left(T\right)-\lambda_i DG_i\left(T\right)\right]\right)+\mathrm{F}\left(S,T;\sum_{j=1}^{m}v_j DH_j\left(T\right)\right)\geqq\ 0. \quad (3.4)$$

(a) From (3.3) that $-\,v_j H_j\left(T\right)\leqq\ 0$, and hence,

$$-\sum_{j=1}^{m}\beta_j\left(S,T\right)v_j H_j\left(T\right)\leqq 0,$$

which by virtue of second part of (i) implies that

$$\mathrm{F}\left(S,T;\sum_{j=1}^{m}v_j DH_j\left(T\right)\right)\leqq-\sigma\, d^2\left(\theta\left(S,T\right)\right). \quad (3.5)$$

From (3.4) and (3.5), we see that

$$\mathrm{F}\left(S,T;\sum_{i=1}^{p}u_i\left[DF_i\left(T\right)-\lambda_i DG_i\left(T\right)\right]\right)\geqq\sigma\, d^2\left(\theta\left(S,T\right)\right)\geqq-\rho d^2\left(\theta\left(S,T\right)\right),$$

where the second inequality follows from (ii). By first part of (i), the last inequality implies that

$$\sum_{i=1}^{p}\alpha_i\left(S,T\right)u_i\left[F_i\left(S\right)-\lambda_i G_i\left(S\right)\right]\geqq\sum_{i=1}^{p}\alpha_i\left(S,T\right)u_i\left[F_i\left(T\right)-\lambda_i G_i\left(T\right)\right]$$

which in view of (3.2) becomes

$$\sum_{i=1}^{p}\alpha_i\left(S,T\right)u_i\left[F_i\left(S\right)-\lambda_i G_i\left(S\right)\right]\geqq 0. \quad (3.6)$$

Since $u_i\alpha_i\left(S,T\right)>0$ for each $i\in\underline{p}$, (3.6) implies that $(F_1(S)-\lambda_1 G_1(S),\ldots,F_p(S)-\lambda_p G_p(S))\nleq(0,\ldots,0)$, which in turn implies that

$$\phi\left(S\right)\equiv\left(\frac{F_1\left(S\right)}{G_1\left(S\right)},\frac{F_2\left(S\right)}{G_2\left(S\right)},\ldots,\frac{F_p\left(S\right)}{G_p\left(S\right)}\right)\nleq\lambda.$$

Proofs for part (b) and (c) are similar to that of part (a). □

Theorem 3.2 *Strong Duality. Let S^*be a regular efficient solution of (P), let* $F(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \left\langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle$ *for any differentiable function* $F: \Lambda^n \to R$ *and* $S \in \Lambda^n$*, and assume that ant one of the three sets of hypotheses specified in Theorem 3.1 holds for all feasible solutions of (DI). Then there exist* $u^* \in U$ *and* $v^* \in R_+^m$ *such that* (S^*, u^*, v^*) *is an efficient solution of (DI) and the objective values of (P) and (DI) are same.*

Proof By Theorem 2.1, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DI). That it is an efficient solution follows from Theorem 3.1. □

4 Dual model II

In this section, we formulate a relatively more general parametric dual model by making use of the partitioning scheme introduced as follows:

Let $\left\{J_0, J_1, \ldots, J_q\right\}$ be a partition of the index set $\underline{m}$; thus, $J_r \subset \underline{m}$ for each $r \in \{0, 1, \ldots, q\}$, $J_r \cap J_s = \phi$ for each $r, s \in \{0, 1, \ldots, q\}$ with $r \neq s$, and $\bigcup_{r=0}^{q} J_r = \underline{m}$.

The duality model considered in this section has the form:

$$\text{Maximize}\, \lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_p\right) \tag{DII}$$

subject to

$$F\left(S, T; \sum_{i=1}^{p} u_i \left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0, \forall\, S \in \Lambda^n, \tag{4.1}$$

$$u_i \left[F_i(T) - \lambda_i G_i(T) + \sum_{j \in J_0} v_j H_j(T)\right] \geqq 0, \; i \in \underline{p}, \tag{4.2}$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0, \; t \in \underline{m}, \tag{4.3}$$

$$T \in \Lambda^{\mathrm{n}}, \; \lambda \in R_+^p, \; u \in U, \; v \in R_+^m,$$

where $F(S, T; \cdot): L_1^n(X, \Lambda, \mu) \to R$ is a sublinear function.

We show that (DII) is a dual problem for (P) by establishing weak and strong duality theorems. In this section, we also use the notations $\Gamma_i(\cdot; \lambda, u, v) = u_i \Big[F_i(S) - \lambda_i G_i(S) + \sum_{j \in J_0} v_j H_j(S)\Big]$, $i \in \underline{p}$ and $\Delta_t(S, v) = \sum_{j \in J_t} v_j H_j(S)$, $t \in \underline{m}$.

Theorem 4.1 *Weak Duality. Let S and* (T, λ, u, v) *be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $\left(\Gamma_i(\cdot\,; \lambda, u, v), \; \Delta_j(\cdot, v)\right)$ $\forall\, i \in \underline{p}$ *and* $j \in \underline{m}$, *are* $(F, \alpha, \beta, \rho, \sigma, \theta)$*-V-pseudo-quasi-type-I at* T;
(ii) $\rho + \sigma \geqq 0$;

(b) *(i)* $\left(\Gamma_i(\cdot\,; \lambda, u, v), \; \Delta_j(\cdot, v)\right)$ $\forall\, i \in \underline{p}$ *and* $j \in \underline{m}$, *are* $(F, \alpha, \beta, \rho, \sigma, \theta)$*-V-pseudo-prestrict-quasi-type-I at* T;
(ii) $\rho + \sigma > 0$;

(c) *(i)* $\left(\Gamma_i(\cdot\,;\lambda,u,v),\ \Delta_j(\cdot,v)\right)$ $\forall\, i \in \underline{p}$ *and* $j \in \underline{m}$, *are* (F, α, β, ρ, σ, θ)*-V-prestrict-quasi-strict-pseudo-type-I at* T;
(ii) $\rho + \sigma \geqq 0$.

Then, $\phi(S) \equiv \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \lambda$.

Proof Let S be an arbitrary feasible solution of (P). Then by the sublinearity of F and (4.1) it follows that

$$\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i\left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{j\in J_0} v_j DH_j(T)\right) + \mathrm{F}\left(S, T; \sum_{t=1}^{m}\sum_{j\in J_t} v_j DH_j(T)\right) \geqq 0. \tag{4.4}$$

(a) Since $v \geqq 0$, $S \in \Xi$ it follows from (4.3) that for each $t \in \underline{m}$:

$$-\sum_{t\in J_t} v_t H_t(T) = -\Delta_t(T,\ v) \leqq 0,$$

and so

$$-\sum_{t=1}^{q} \beta_t(S,T)\Delta_t(T,v) \leqq 0,$$

which by virtue of second part of (i) implies that

$$\mathrm{F}\left(S, T; \sum_{t=1}^{q}\sum_{j\in J_t} v_j DH_j(T)\right) \leqq -\sigma\, d^2(\theta(S,T)). \tag{4.5}$$

From (4.4) and (4.5), we see that

$$\begin{aligned}\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i\left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{j\in J_0} v_j DH_j(T)\right) &\geqq \sigma\, d^2(\theta(S,T)) \\ &\geqq -\rho\, d^2(\theta(S,T)),\end{aligned}$$

where the second inequality follows from (ii). By virtue of the first part of hypothesis (i), the above inequality implies that

$$\sum_{i=1}^{p} \alpha_i(S,T)\,\Gamma_i(\mathrm{S},\lambda,u,v) \geqq \sum_{i=1}^{p} \alpha_i(S,T)\,\Gamma_i(\mathrm{T},\lambda,u,v). \tag{4.6}$$

Since $\alpha_i(S,T) > 0$, $u_i \geqq 0$, $\forall i \in \underline{p}$, and (4.2) holds, we deduce from (4.6) that

$$\sum_{i=1}^{p} \alpha_i(S,T)\,\Gamma_i(\mathrm{S},\lambda,u,v) \geqq 0,$$

which simplifies to

$$\sum_{i=1}^{p} \alpha_i(S,T)\,u_i\left[F_i(S) - \lambda_i G_i(S)\right] \geqq 0.$$

which is precisely (3.6). Therefore, the rest of the proof is identical to that of Part (a) of Theorem 3.1.

Proofs of parts (b) and (c) are similar to that of part (a). □

Remark 4.1 Note that Theorem 4.1 contains a number of special cases that can easily be identified by appropriate choices of the partitioning sets $J_0, J_1, \ldots, J_q$.

Theorem 4.2 *Strong Duality. Let* S^**be a regular efficient solution of (P), let* $Ғ(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \left\langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle$ *for any differentiable function* $F: \Lambda^n \to R$ *and* $S \in \Lambda^n$*, and assume that ant one of the three sets of hypotheses specified in Theorem 4.1 holds for all feasible solutions of (DII). Then there exist* $u^* \in U$ *and* $v^* \in R_+^m$ *such that* (S^*, u^*, v^*) *is an efficient solution of (DII) and the objective values of (P) and (DII) are same.*

Proof By Theorem 2.1, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DII). That it is an efficient solution follows from Theorem 4.1. □

5 Dual model III

In this section, we present another general parametric dual model for (P). It is again based on the partitioning scheme employed in the previous section. The dual model can be stated as follows:

$$\text{Maximize}\,\lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_p\right) \tag{DIII}$$

subject to

$$Ғ\left(S, T; \sum_{i=1}^{p} u_i \left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0, \forall S \in \Lambda^n, \tag{5.1}$$

$$F_i(T) - \lambda_i G_i(T) \geqq 0, \ i \in \underline{p}, \tag{5.2}$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0, \ t \in \underline{m} \cup \{0\}, \tag{5.3}$$

$$T \in \Lambda^{\rm n}, \lambda \in R_+^p, \ u \in U, \ v \in R_+^m,$$

where $Ғ(S, T; \cdot): L_1^n(X, \Lambda, \mu) \to R$ is a sublinear function.

We show that (DIII) is a dual problem for (P) by establishing weak and strong duality theorems. Let $\{I_0, I_1, \ldots, I_k\}$ be a partition of $\underline{p}$ such that $K = \{0, 1, \ldots, k\} \subset Q = \{0, 1, \ldots, q\}$, $k < q$,and let the function $\Theta_t(\cdot, \lambda, u, v): \Lambda^n \to R$ be defined for fixed λ, u and v by

$$\Theta_t(S, \lambda, u, v) = \sum_{i \in I_t} u_i \left[F_i(S) - \lambda_i G_i(S)\right] + \sum_{j \in J_t} v_j H_j(S), \quad t \in K.$$

and $\Delta_t(S, v) = \sum_{j \in J_t} v_j H_j(S), \quad t \in \underline{m}$.

Theorem 5.1 *Weak Duality. Let S and* (T, λ, u, v) *be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $\left(\Theta_t(\cdot, \lambda, u, v),\ \Delta_j(\cdot, v)\right)$ $\forall\, t \in K$ *and* $j \in \underline{m},$ *are* $(\mathrm{F}, \alpha, \beta, \rho, \sigma, \theta)$*-V-strict pseudo-quasi-type-I at* T;
(ii) $\rho + \sigma \geqq 0$;

(b) *(i)* $\left(\Theta_t(\cdot, \lambda, u, v),\ \Delta_j(\cdot, v)\right)$ $\forall\, t \in K$ *and* $j \in \underline{m},$ *are* $(\mathrm{F}, \alpha, \beta, \rho, \sigma, \theta)$*-V-pseudo-prestrict-quasi-type-I at* T;
(ii) $\rho + \sigma > 0$;

(c) *(i)* $\left(\Theta_t(\cdot, \lambda, u, v),\ \Delta_j(\cdot, v)\right)$ $\forall\, t \in K$ *and* $j \in \underline{m},$ *are* $(\mathrm{F}, \alpha, \beta, \rho, \sigma, \theta)$*-V-pre-strict-quasi-strict-pseudo-type-I at* T;
(ii) $\rho + \sigma \geqq 0$.

Then, $\phi(S) \equiv \left(\dfrac{F_1(S)}{G_1(S)}, \dfrac{F_2(S)}{G_2(S)}, \ldots, \dfrac{F_p(S)}{G_p(S)}\right) \nleq \lambda$.

Proof Suppose to the contrary that $\phi(S) \leq \lambda$. This implies that $F_i(S) - \lambda_i G_i(S) \leqq 0, \forall\, i \in \underline{p}$, with strict inequality holding for at least one $l \in \underline{p}$. From these inequalities, non-negativity of v, primal feasibility of S, and (5.2) it is easily seen that for each $t \in K$,

$$\begin{aligned}\Theta_t(S, \lambda, u, v) &= \sum_{i \in I_t} u_i\left[F_i(S) - \lambda_i G_i(S)\right] + \sum_{j \in J_t} v_j H_j(S)\\ &\leqq \sum_{i \in I_t} u_i\left[F_i(S) - \lambda_i G_i(S)\right]\\ &\leqq 0\\ &= \sum_{i \in I_t} u_i\left[F_i(S) - \lambda_i G_i(S)\right] + \sum_{j \in J_t} v_j H_j(S) = \Theta_t(S, \lambda, u, v)\end{aligned}$$

and so

$$\sum_{t \in K} \alpha_t(S, T)\Theta_t(S, \lambda, u, v) < \sum_{t \in K} \alpha_t(S, T)\Theta_t(T, \lambda, u, v),$$

which in view of first part of the hypotheses (i) implies that

$$\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i\left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{t \in K}\sum_{j \in J_t} v_j DH_j(T)\right) < -\rho\, d^2(\theta(S, T)). \quad (5.4)$$

As for each $t \in M \backslash K$, $-\sum_{t \in M \backslash K} \beta_t(S, T)\, \Delta_t(S, v) \leqq 0$, and hence the second part of the hypotheses a (i) implies that

$$\mathrm{F}\left(S, T; \sum_{t \in M \backslash K}\sum_{j \in J_t} v_j DH_j(T)\right) \leqq -\sigma\, d^2(\theta(S, T)). \quad (5.5)$$

Now from (5.4), (5.5), a (ii) and sublinearity, we get

$$\begin{aligned}\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i\left[DF_i(T) - \lambda_i DG_i(T)\right] + \sum_{j=1}^{m} v_j DH_j(T)\right) &< -(\rho + \sigma)\, d^2(\theta(S, T))\\ &< 0,\end{aligned}$$

which contradicts (5.1). Hence, $\phi(S) \equiv \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \lambda$.
Proofs of parts (b) and (c) are similar to that of part (a). □

Theorem 5.2 *Strong Duality. Let S^* be a regular efficient solution of (P), let* $\digamma(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \left\langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*} \right\rangle$ *for any differentiable function $F: \Lambda^n \to R$ and $S \in \Lambda^n$, and assume that any one of the three sets of hypotheses specified in Theorem 5.1 holds for all feasible solutions of (DIII). Then there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an efficient solution of (DIII) and the objective values of (P) and (DIII) are same.*

Proof By Theorem 2.1, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DIII). That it is an efficient solution follows from Theorem 5.1. □

6 Dual model IV

In this section, we investigate the following dual model for (P), which may be written as the semi-parametric counterpart of (DI):

$$\text{Maximize } \left(\frac{F_1(T)}{G_1(T)}, \ldots, \frac{F_p(T)}{G_p(T)}\right) \tag{DIV}$$

subject to

$$\digamma\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T) DF_i(T) - F_i(T) DG_i(T)\right] + \sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0, \forall S \in \Lambda^n, \tag{6.1}$$

$$v_j H_j(T) \geqq 0, \ t \in \underline{m}, \tag{6.2}$$

$$T \in \Lambda^n, \ u \in U, \ v \in R_+^m,$$

where $\digamma(S, T; \cdot): L_1^n(X, \Lambda, \mu) \to R$ is a sublinear function. In the remaining part of this paper, we assume that $G_i(T) > 0$ and $F_i(T) \geqq 0, i \in \underline{p}$, for all T and u such that (T, u, v) is a feasible solution of the dual problem under consideration. In addition, in the statements and proofs of theorems to follow in this section, we use the notations, $E_i(\cdot, T, u)$, $B_j(\cdot, v)$ and $L_i(\cdot, T, u, v)$ defined for fixed $S, u,$ and v on Λ^n by

$$E_i(S, T, u) = u_i \left\lfloor G_i(T) F_i(S) - F_i(T) G_i(S) \right\rceil, \quad \forall i \in \underline{p},$$
$$B_i(S, v) = v_j H_j(S), \ j \in \underline{m},$$

and

$$L_i(S, T, u, v) = u_i \left[G_i(T) F_i(S) - F_i(T) G_i(S) + \sum_{j \in J_0} v_j H_j(S)\right], i \in \underline{p}.$$

Now we establish weak, strong and strict converse duality theorem for (P) and (DIV).

Theorem 6.1 *Weak Duality. Let S and (T, u, v) be arbitrary feasible solutions for (P) and (DIV), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $\left(\mathrm{E}_i,\ B_j\right)$, $\forall i \in \underline{p}$, *and* $\forall\ j \in \underline{m}$ *are* $(\mathrm{F}, \alpha, \beta, \rho, \sigma, \theta)$*-V-pseudo-quasi-type-I at* T;
(ii) $\rho + \sigma \geqq 0$;

(b) *(i)* $\left(\mathrm{E}_i\,(\cdot\,, T, u)\,,\ \mathrm{B}_j\,(\cdot, v)\right)$ $\forall\ i \in \underline{p}$ *and* $j \in \underline{m}$, *are* $(\mathrm{F}, \alpha, \beta, \rho, \sigma, \theta)$*-V-pseudo-pre-strict-quasi-type-I at* T;
(ii) $\rho + \sigma > 0$;

(c) *(i)* $\left(\mathrm{E}_i\,(\cdot\,, T, u)\,,\ \mathrm{B}_j\,(\cdot, v)\right)$ $\forall\, i \in \underline{p}$ *and* $j \in \underline{m}$, *are* $(\mathrm{F}, \alpha, \beta, \rho, \sigma, \theta)$*-V-prestrict-quasi-strict-pseudo-type-I at* T;
(ii) $\rho + \sigma \geqq 0$.

Then, $\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)}{G_1(T)}, \frac{F_2(T)}{G_2(T)}, \ldots, \frac{F_p(T)}{G_p(T)}\right)$.

Proof Let S be an arbitrary feasible solution of (P). Then by the sublinearity of F and (6.1), it follows that

$$\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i\left[G_i(T)\, DF_i(T) - F_i(T)\, DG_i(T)\right]\right) + \mathrm{F}\left(S, T; \sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0. \quad (6.3)$$

Following as in the proof of Theorem 3.1, from the second part of the assumption a (i) and (6.3), we get

$$\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i\left[G_i(T)\, DF_i(T) - F_i(T)\, DG_i(T)\right]\right) \geqq -\rho\, d^2\left(\theta(S, T)\right),$$

which in the light of the hypotheses implies that

$$\sum_{i=1}^{p} \alpha_i(S, T)\, u_i\left[G_i(T)\, F_i(S) - F_i(T)\, G_i(S)\right] \geqq \sum_{i=1}^{p} \alpha_i(S, T)\, u_i\left[G_i(T)\, F_i(T) - F_i(T)\, G_i(T)\right] = 0. \quad (6.4)$$

Since $\alpha_i(S, T)\, u_i > 0$ for each $i \in \underline{p}$, (6.4) implies that $\left(G_1(T)\,\mathrm{F}_1(S) - \mathrm{F}_1(T)\, G_1(S), \ldots, G_p(T)\,\mathrm{F}_p(S) - \mathrm{F}_p(T)\, G_p(S)\right) \nleq (0, \ldots, 0)$, which in turn implies that

$$\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)}{G_1(T)}, \frac{F_2(T)}{G_2(T)}, \ldots, \frac{F_p(T)}{G_p(T)}\right).$$

Proofs of parts (b) and (c) are similar to that of part (a). □

Theorem 6.2 *Strong Duality. Let* S^* *be a regular efficient solution of (P), let* $\mathrm{F}(S, S^*; DF(S^*)) = \sum_{k=1}^{n}\left\langle D_k F(S^*),\ \chi_{S_k} - \chi_{S_k^*}\right\rangle$ *for any differentiable function* $F\colon \Lambda^n \to R$ *and* $S \in \Lambda^n$, *and assume that ant one of the three sets of hypotheses specified in Theorem 6.1 holds for all feasible solutions of (DIV). Then there exist* $u^* \in U$ *and* $v^* \in R_+^m$ *such that* (S^*, u^*, v^*) *is an efficient solution of (DIV) and the objective values of (P) and (DIV) are same.*

Proof By Theorem 2.2, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DIV). That it is an efficient solution follows from Theorem 6.1. □

7 Dual model V

In this section, we present a more general semi-parametric dual model for (P):

$$\text{Maximize}\left(\frac{F_1(T)+\sum_{j\in J_0} v_j H_j(T)}{G_1(T)},\ldots,\frac{F_p(T)+\sum_{j\in J_0} v_j H_j(T)}{G_p(T)}\right) \tag{DV}$$

subject to

$$\text{Ғ}\left(S,T;\sum_{i=1}^{p} u_i\left[\begin{array}{l} G_i(T)\left[DF_i(T)+\sum\limits_{j\in J_0} v_j DH_j(T)\right] \\ -\left[F_i(T)+\Delta_0(T,v)\right]DG_i(T)\end{array}\right]+\sum_{j\in \underline{m}\setminus J_0} v_j DH_j(T)\right)\geqq 0,\quad \forall\, S\in\Lambda^n,$$

$$\sum_{j\in J_t} v_j H_j(T)\geqq 0,\ t\in \underline{m}\cup\{0\}, \tag{7.1}$$

$$T\in\Lambda^{\mathrm{n}},\ u\in U,\ v\in R_+^m, \tag{7.2}$$

where $\text{Ғ}(S,T;\ \cdot):L_1^n(X,\Lambda,\mu)\to R$ is a sublinear function. In addition, in the statements and proofs of theorems to follow in this section, we use the following notation defined for fixed S, u, and v on Λ^n by:

$$\Pi_i(S,T,u,v)=u_i\left[G_i(T)\left\{F_i(S)+\sum_{j\in J_0} v_j DH_j(S)\right\}-\{F_i(T)+\Delta_0(T,v)\}G_i(S)\right],\quad \forall, i\in\underline{p}$$

Theorem 7.1 *Weak Duality. Let Sand $(T,\ u,\ v)$ be arbitrary feasible solutions for (P) and (DV), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) *(i)* $(\Pi_i(\cdot,T,v),\Delta_j(\cdot,v))$, $\forall i\in\underline{p}$, *and* $\forall j\in\underline{m}$ *are* (Ғ, α, β, ρ, σ, θ)*-V-pseudo-quasi-type-I at* T;
(ii) (ii) $\rho+\sigma\geqq 0$;

(b) *(i)* $(\Pi_i(\cdot,T,v),\Delta_j(\cdot,v))$, $\forall i\in\underline{p}$, *and* $\forall j\in\underline{m}$ *are* (Ғ, α, β, ρ, σ, θ)*-V-pseudo-prestrict-quasi-type-I at* T;
(ii) $\rho+\sigma>0$;

(c) *(i)* $(\Pi_i(\cdot,T,v),\Delta_j(\cdot,v))$, $\forall i\in\underline{p}$, *and* $\forall j\in\underline{m}$ *are* (Ғ, α, β, ρ, σ, θ)*-V-pre-strict-quasi-strict-pseudo-type-I at* T;
(ii) $\rho+\sigma\geqq 0$.

Then,

$$\left(\frac{F_1(S)}{G_1(S)},\frac{F_2(S)}{G_2(S)},\ldots,\frac{F_p(S)}{G_p(S)}\right)\nleq\left(\frac{F_1(T)+\sum_{j\in J_0} v_j H_j(T)}{G_1(T)},\frac{F_2(T)+\sum_{j\in J_0} v_j H_j(T)}{G_2(T)},\ldots,\frac{F_p(T)+\sum_{j\in J_0} v_j H_j(T)}{G_p(T)}\right).$$

Proof (a) Let S be an arbitrary feasible solution of (P). Then by the sublinearity of F and (7.1), it follows that

$$\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T)\left\{DF_i(T)+\sum_{j\in J_0} v_j DH_j(T)\right\} - \{F_i(T)+\Delta_0(T, v)\} DG_i(T)\right]\right) + \mathrm{F}\left(S, T; \sum_{t=1}^{m}\sum_{j\in J_t} v_j DH_j(T)\right) \geqq 0. \tag{7.3}$$

Following as in the proof of Theorem 3.1, from the second part of the assumption a (i) and (7.3), we get

$$\mathrm{F}\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T)\left\{DF_i(T)+\sum_{j\in J_0} v_j DH_j(T)\right\} - \{F_i(T) + \Delta_0(T, v)\} DG_i(T)\right]\right) \geqq -\rho\, d^2(\theta(S, T)),$$

which in the light of the hypotheses implies that

$$\sum_{i=1}^{p} \alpha_i(S, T)\,\Pi_i(S, T, u, v) \geqq \sum_{i=1}^{p} \alpha_i(S, T)\,\Pi_i(T, T, u, v) = 0. \tag{7.4}$$

The equality holds due to the fact that $\Pi_i(T, T, u, v) = 0$. Since $\alpha_i(S, T)\, u_i > 0$ for each $i \in \underline{p}$,(7.4) implies that

$$\big(G_1(T)F_1(S) - [F_1(T) + \Delta_0(T, v)]G_1(S), \ldots, G_p(T)F_p(S) - \lfloor F_p(T) + \Delta_0(T, v)\rfloor G_p(S)\big) \nleq (0, \ldots, 0),$$

which in turn implies that

$$\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)+\sum_{j\in J_0} v_j H_j(T)}{G_1(T)}, \frac{F_2(T)+\sum_{j\in J_0} v_j H_j(T)}{G_2(T)}, \ldots, \frac{F_p(T)+\sum_{j\in J_0} v_j H_j(T)}{G_p(T)}\right).$$

Proofs of parts (b) and (c) are similar to that of part (a). □

Theorem 7.2 *Strong Duality. Let* S^* *be a regular efficient solution of (P), let* $\mathrm{F}(S, S^*; DF(S^*)) = \sum_{k=1}^{n}\left\langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*}\right\rangle$ *for any differentiable function* $F: \Lambda^n \to R$ *and* $S \in \Lambda^n$*, and assume that any one of the three sets of hypotheses specified in Theorem 7.1 holds for all feasible solutions of (DV). Then there exist* $u^*\in U$ *and* $v^*\in R_+^m$ *such that* (S^*, u^*, v^*) *is an efficient solution of (DV) and the objective values of (P) and (DV) are same.*

Proof By Theorem 2.2, there exist $u^*\in U$ and $v^*\in R_+^m$ such that (S^*, u^*, v^*) is a feasible solution of (DV), using the arguments as in the proof of Theorem 9.2 [42]. That it is an efficient solution follows from Theorem 7.1. □

8 Dual model VI

In this section, we discuss another general dual model for (P) which may be viewed as the semi-parametric version of (DIII). It can be stated as follows:

$$\text{Maximize } \left(\frac{F_1(T)}{G_1(T)}, \ldots, \frac{F_p(T)}{G_p(T)}\right) \tag{DVI}$$

subject to

$$Ƒ\left(S, T; \sum_{i=1}^{p} u_i \left[G_i(T) DF_i(T) - F_i(T) DG_i(T)\right] + \sum_{j=1}^{m} v_j DH_j(T)\right) \geqq 0, \forall S \in \Lambda^n, \tag{8.1}$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0, \; t \in \underline{m} \cup \{0\}, \tag{8.2}$$

$$T \in \Lambda^{\mathrm{n}}, \; u \in U, \; v \in R_+^m,$$

where $Ƒ(S, T; \cdot) : L_1^n(X, \Lambda, \mu) \to R$ is a sublinear function.

Next, we shall show that (DVI) is a dual problem to (P) by proving weak and strong duality theorems.

Theorem 8.1 *Weak Duality. Let S and (T, u, v) be arbitrary feasible solutions for (P) and (DVI), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $\left(\Pi_i(\cdot, T, v), \Delta_j(\cdot, v)\right)$, $\forall i \in \{1, \ldots, k\}$ and $\forall j \in \{k+1, \ldots, m\}$ are (Ƒ, α, β, ρ, σ, θ)-V-pseudo-quasi-type-I at T;
(ii) (ii) $\rho + \sigma \geqq 0$;

(b) (i) $\left(\Pi_i(\cdot, T, v), \Delta_j(\cdot, v)\right)$, $\forall i \in \{1, \ldots, k\}$ and $\forall j \in \{k+1, \ldots, m\}$ are (Ƒ, α, β, ρ, σ, θ)-V-pseudo-prestrict-quasi-type-I at T;
(ii) $\rho + \sigma > 0$;

(c) (i) $\left(\Pi_i(\cdot, T, v), \Delta_j(\cdot, v)\right)$, $\forall i \in \{1, \ldots, k\}$ and $\forall j \in \{k+1, \ldots, m\}$ are (Ƒ, α, β, ρ, σ, θ)-V-prestrict-quasi-strict-pseudo-type-I at T;
(ii) $\rho + \sigma \geqq 0$.

Then, $\left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \ldots, \frac{F_p(S)}{G_p(S)}\right) \nleq \left(\frac{F_1(T)}{G_1(T)}, \frac{F_2(T)}{G_2(T)}, \ldots, \frac{F_p(T)}{G_p(T)}\right)$.

Proof The proof can be done following the discussions above in this paper and the proof of the Theorem 10.1 [42]. □

Theorem 8.2 *Strong Duality. Let S^*be a regular efficient solution of (P), let $Ƒ(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \left\langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*}\right\rangle$ for any differentiable function $F: \Lambda^n \to R$ and $S \in \Lambda^n$, and assume that any one of the three sets of hypotheses specified in Theorem 8.1 holds for all feasible solutions of (DVI). Then there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an efficient solution of (DVI) and the objective values of (P) and (DVI) are same.*

Proof By Theorem 2.2, there exist $u^* \in U$ and $v^* \in R_+^m$ such that (S^*, u^*, v^*) is an feasible solution of (DVI). That it is an efficient solution follows from Theorem 8.1. □

9 Conclusion

In this paper, we have established various duality theorems for (P) and six different types of dual models to (P) under generalized (F, α, β, ρ, σ, θ)-V-type-I and related non-convex functions for a multiobjective fractional subset programming problem. These duality results are extension of corresponding results to the case of more general class of functions as compared to that of Zalmai [41]. This work can be further extended to the class of functions introduced recently, by Hachimi and Aghezzaf [6].

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